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Stationary strings and branes in the higher-dimensional Kerr-NUT-(A)dS spacetimes

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Abstract: We demonstrate complete integrability of the Nambu-Goto equations for a stationary string in the general Kerr-NUT-(A)dS spacetime describing the higher-dimensional rotating black hole. The stationary string in D dimensions is generated by a 1-parameter family of Killing trajectories and the problem of finding a string configuration reduces to a problem of finding a geodesic line in an effective $(D-1)$-dimensional space. Resulting integrability of this geodesic problem is connected with the existence of hidden symmetries which are inherited from the black hole background. In a spacetime with p mutually commuting Killing vectors it is possible to introduce a concept of a ξ-brane, that is a p-brane with the worldvolume generated by these fields and a 1-dimensional curve. We discuss integrability of such ξ-branes in the Kerr-NUT-(A)dS spacetime.

Keywords: Large Extra Dimensions, Black Holes, p-branes.

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Contents

1. Introduction

There are several reasons why the problem of interaction of strings and branes with black holes attracted interest recently. Fundamental strings and branes are basic objects in string theory [1], and black holes (as well as other black objects) form an important class of solutions of the low-energy effective action in this theory (see, e.g., [2]). On the other hand, cosmic strings and domain walls are topological defects which can be naturally created during phase transitions in the early Universe (see, e.g., [3 – 5]). Their interaction with astrophysical black holes may result in interesting observational effects. In both cases we are dealing with a problem when the interacting objects are non-local and relativistic. An important example is an interaction of a bulk black hole with a brane representing our world in the brane world models (see, e.g., [6]). A stationary test brane interacting with a bulk black hole can be used as a toy model for the study of (Euclidean) topology change transitions [7]. This model demonstrates interesting scaling and self-similarity properties during such phase transitions, similar to the Choptuik critical collapse [8] and merger black hole transitions [9]. These models may also have far going interesting consequences for the study of phase transitions in quantum chromodynamics (see, e.g., [10, 11]).

Even in an idealized case, when one neglects the effects connected with the thickness of the strings and branes and their tension, this problem is quite complicated. The reason is evident: the Dirac-Nambu-Goto action for these objects in an external gravitational field is very nonlinear. In a general case numerical calculations are required (see, e.g., [12]). When the effects of thickness and tension are taken into account these numerical calculations become even more involved (see, e.g., [13, 14]).

Study of *stationary* configurations of strings and branes in a background of a stationary black hole is simpler problem which in several cases allows complete solution. One of the examples is a stationary string in the Kerr spacetime. It was shown [15] that the Hamilton-Jacobi equation for such a string allows a complete separation of variables. It was also demonstrated [16] that this property is directly connected with the hidden symmetry of the Kerr metric generated by the Killing tensor [17] discovered by Carter in 1968 [18]. More recently, Carters's method was applied to 5-dimensional rotating black holes and the Killing tensor was found in these spacetimes [19]. This result was used to show that the equations for a stationary string in the 5-dimensional Myers-Perry metric are completely integrable [20].

In the present paper we demonstrate that this result allows a generalization to higher-dimensional rotating black holes in an arbitrary number of spacetime dimensions. Namely, we show that a stationary string configuration is completely integrable in the general Kerr-NUT-(A)dS spacetimes [21]. We use the fact that after performing a dimensional reduction along the Killing trajectories, the stationary string equation in a D-dimensional stationary spacetime can be reduced to a geodesic equation in a $(D-1)$-dimensional space with a metric conformal to the reduced metric. The separability of the Hamilton-Jacobi equation in this effective metric follows from the separability of the Hamilton-Jacobi equation in the original D-dimensional Kerr-NUT-(A)dS spacetime proved in [22] and a special property of the primary (timelike) Killing vector.

There is a natural generalization of the concept of a stationary string in the case when there exist several mutually commuting Killing vectors. If p is a number of these fields one may consider a $(p+1)$-hypersurface generated by the Killing vectors passing through a 1-dimensional line. We call a ξ-brane an extended object, a p-brane, with the worldvolume associated with this hypersurface. We discuss integrability conditions for ξ-branes in the Kerr-NUT-(A)dS spacetimes [21] and give some examples of integrable systems.

2. Stationary strings

Consider a string in a stationary D-dimensional spacetime M^D. Let x^a $(a = 0, \ldots, D-1)$ be coordinates in it and

$$ds^2 = g_{ab}dx^a dx^b \tag{2.1}$$

be its metric. We denote by ξ^a the corresponding Killing vector which is timelike at least in some domain of M^D. We call the string *stationary* if ξ^a is tangent to the 2-dimensional worldsheet Σ_ξ of the string in this domain. In other words, the surface Σ_ξ is generated by a 1-parameter family of the Killing trajectories (the integral lines of ξ^a).

A general formalism for studying a stationary spacetime based on its foliation by Killing trajectories was developed by Geroch [23]. In this approach, one considers a set S of the Killing trajectories as a quotient space and introduce the structure of the differential

Riemannian manifold on it. The projector h_{ab} onto S is related to the metric g_{ab} as follows

$$g_{ab} = h_{ab} + \xi_a \xi_b / \xi^2 \,. \tag{2.2}$$

In this formalism, a stationary string is uniquely determined by a curve in S.

The equation for this curve follows from the Nambu-Goto action

$$I = -\mu \int d^2\zeta \, |\gamma|^{1/2} \,. \tag{2.3}$$

Here μ is the string tension. As it enters the Nambu-Goto action as a common factor, its value is not important and one can always put $\mu = 1$. The string worldsheet can be parametrized by $x^a = x^a(\zeta^A)$, where ζ^A are coordinates on Σ_ξ, $(A = 0, 1)$. We denote by γ_{AB} the induced metric on Σ_ξ

$$\gamma_{AB} = \frac{\partial x^a}{\partial \zeta^A} \frac{\partial x^a}{\partial \zeta^B} g_{ab} \,, \tag{2.4}$$

and by γ its determinant.

Let Killing time parameter be t, so that $\xi^a \partial_a = \partial_t$, and let y^i be coordinates which are constant along the Killing trajectories (coordinates in S). Then, the non-vanishing components of the projection operator h_{ab} are h_{ij} (reduced metric) and the metric (2.1)-(2.2) takes the form

$$ds^2 = -F(dt + A_i dy^i)^2 + h_{ij} dy^i dy^j \,, \tag{2.5}$$

$$F = g_{tt} = -\xi_a \xi^a \,, \qquad A_i = g_{ti}/g_{tt} \,. \tag{2.6}$$

From (2.2) it also follows that in these coordinates $h^{ij} = g^{ij}$.

We choose $\zeta^0 = t$ and denote $\zeta^1 = \sigma$. Then the string configuration is determined by $y^i = y^i(\sigma)$. The induced metric is

$$d\gamma^2 = \gamma_{AB} d\zeta^A d\zeta^B = -F(dt + A d\sigma)^2 + dl^2 \,, \tag{2.7}$$

where

$$dl^2 = h d\sigma^2 \,, \; A = A_i dy^i/d\sigma \,, \; h = h_{ij} \frac{dy^i}{d\sigma} \frac{dy^j}{d\sigma} \,, \tag{2.8}$$

and it has the following determinant

$$\gamma = \det(\gamma_{AB}) = -F h \,. \tag{2.9}$$

So, the Nambu-Goto action is

$$I = -\Delta t E \,, \tag{2.10}$$

$$E = \mu \int \sqrt{F} dl = \mu \int d\sigma \sqrt{F h_{ij} \frac{dy^i}{d\sigma} \frac{dy^j}{d\sigma}} \,. \tag{2.11}$$

In a static spacetime the equation (2.11) has a very simple meaning: The energy density of a string is proportional to its proper length dl multiplied by the red-shift factor $\sqrt{F}$.

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The problem of a stationary string configuration therefore reduces to that of a geodesic in the $(D-1)$-dimensional effective background

$$dH^2 = H_{ij}dy^i dy^j = F h_{ij} dy^i dy^j \,. \tag{2.12}$$

In order to solve this geodesic problem we shall use the Hamilton-Jacobi method. That is, we shall attempt for the additive separation of the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial \sigma} + H^{ij}\, \partial_i S\, \partial_j S = 0 \,, \tag{2.13}$$

where H^{ij} is the inverse of the effective metric (2.12) with the components given by

$$F H^{ij} = h^{ij} = g^{ij}. \tag{2.14}$$

If the Hamilton-Jacobi equation can be separated, the effective geodesic motion and hence also the stationary string configuration are completely integrable, e.g., [24].

3. Stationary strings in Kerr-NUT-AdS spacetime

In this section we prove the complete integrability of a stationary string configuration in the general Kerr-NUT-(A)dS spacetime [21]. After a suitable analytical continuation the metric takes the form[1]

$$ds^2 = \sum_{\mu=1}^{n} \left[\frac{dx_\mu^2}{Q_\mu} + Q_\mu \Bigg(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \Bigg)^2 \right] - \frac{\varepsilon c}{A^{(n)}} \Bigg(\sum_{k=0}^{n} A^{(k)} d\psi_k \Bigg)^2, \tag{3.1}$$

with $n = [D/2]$ and $\varepsilon = D - 2n$. Here,

$$\begin{aligned}
A_\mu^{(k)} &= \sum_{\substack{\nu_1 < \cdots < \nu_k \\ \nu_i \neq \mu}} x_{\nu_1}^2 \ldots x_{\nu_k}^2, \qquad & A^{(k)} &= \sum_{\nu_1 < \cdots < \nu_k} x_{\nu_1}^2 \ldots x_{\nu_k}^2 \,, \\
Q_\mu &= \frac{X_\mu}{U_\mu} \,, & U_\mu &= \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\nu^2 - x_\mu^2) \,, \\
X_\mu &= \sum_{k=\varepsilon}^{n} c_k x_\mu^{2k} - 2b_\mu x_\mu^{1-\varepsilon} + \frac{\varepsilon c}{x_\mu^2} \,.
\end{aligned} \tag{3.2}$$

Time is denoted by ψ_0, azimuthal coordinates by ψ_k, $k = 1, \ldots, m = D - n - 1$, and x_μ, $\mu = 1, \ldots, n$, stand for radial and latitude coordinates. Parameter c_n is proportional to the cosmological constant [25]

$$R_{ab} = (-1)^n (D-1) c_n \, g_{ab} \,, \tag{3.3}$$

[1] The physical metric with proper signature is recovered when standard radial coordinate $r = -ix_n$ and new parameter $M = (-i)^{1+\epsilon} b_n$ are introduced (for more details see [21]). As these transformations do not affect the discussed separability of the Hamilton-Jacobi equation we prefer to work with this more symmetric analytical continuation of the metric.

and remaining constants c_k, c, and b_μ are related to rotation parameters, mass, and NUT parameters of the black hole (see [21] for more details). The inverse metric reads

$$g^{ab}\partial_a\partial_b = \sum_{\mu=1}^{n}\frac{1}{X_\mu U_\mu}\Big(\sum_{k=0}^{m}(-x_\mu^2)^{n-1-k}\partial_{\psi_k}\Big)^2 + \sum_{\mu=1}^{n}Q_\mu(\partial_{x_\mu})^2 - \frac{\varepsilon}{cA^{(n)}}\,(\partial_{\psi_n})^2\,. \tag{3.4}$$

In the space with the metric (3.1) the vector ∂_{ψ_0}, called *primary Killing*, plays a special role. This vector (after the analytical continuation to the 'physical' spacetime) is timelike in the black hole exterior. It is also directly connected with the principal Killing-Yano tensor of the metric [26]. We call a string stationary if it is tangent to the primary Killing vector. For this string one has

$$H^{ij}\partial_i\partial_j = F^{-1}\Big[\sum_{\mu=1}^{n}\frac{1}{X_\mu U_\mu}\Big(\sum_{k=1}^{m}(-x_\mu^2)^{n-1-k}\partial_{\psi_k}\Big)^2 + \sum_{\mu=1}^{n}Q_\mu(\partial_{x_\mu})^2 - \frac{\varepsilon}{cA^{(n)}}\,(\partial_{\psi_n})^2\Big] \tag{3.5}$$

$$F = \sum_{\mu=1}^{n}Q_\mu - \frac{\varepsilon c}{A^{(n)}}\,. \tag{3.6}$$

The expression in the square brackets of (3.5), the reduced metric, is similar to (3.4). The only difference is that in the sum over k the term $k=0$ is omitted. This corresponds to the natural projection given by (2.14).

In the background of the metric H_{ij} the Hamilton-Jacobi equation (2.13) allows the additive separation of variables

$$S = w\sigma + \sum_{\mu=1}^{n}S_\mu(x_\mu) + \sum_{k=1}^{m}L_k\psi_k \tag{3.7}$$

with functions $S_\mu(x_\mu)$ of a single argument x_μ. Substituting (3.7) into (2.13) we obtain

$$Fw + \sum_{\mu=1}^{n}\frac{1}{X_\mu U_\mu}\Big(\sum_{k=1}^{m}(-x_\mu^2)^{n-1-k}L_k\Big)^2 + \sum_{\mu=1}^{n}Q_\mu S_\mu'^2 - \frac{\varepsilon L_n^2}{cA^{(n)}} = 0\,, \tag{3.8}$$

where ${S_\mu}'$ denotes the derivative of a function S_μ with respect to its single argument x_μ. Using the explicit form of F and algebraic identity [22]:

$$\frac{1}{A^{(n)}} = \sum_{\mu=1}^{n}\frac{1}{x_\mu^2 U_\mu}\,, \tag{3.9}$$

we can rewrite the last equation in the form

$$\sum_{\mu=1}^{n}\frac{G_\mu}{U_\mu} = 0, \tag{3.10}$$

where G_μ are functions of x_μ only:

$$G_\mu = X_\mu\left(S_\mu'^2 + w\right) + \frac{1}{X_\mu}\Big(\sum_{k=1}^{m}(-x_\mu^2)^{n-1-k}L_k\Big)^2 - \frac{\varepsilon\left(L_n^2/c + wc\right)}{x_\mu^2}\,. \tag{3.11}$$

Applying the Lemma proved in the appendix of [27] we realize that the general solution of (3.10) is

$$G_\mu = \sum_{k=1}^{n-1} C_k (-x_\mu^2)^{n-1-k} \,, \tag{3.12}$$

where C_k are arbitrary constants. So, we have obtained the equations for S'_μ:

$$S'^2_\mu = \frac{1}{X_\mu}\left[\sum_{k=1}^{n-1} C_k \,(-x_\mu^2)^{n-1-k} + \frac{\varepsilon\left(L_n^2/c + wc\right)}{x_\mu^2}\right] - \frac{1}{X_\mu^2}\left(\sum_{k=1}^{m}(-x_\mu^2)^{n-1-k} L_k\right)^2 - w\,, \tag{3.13}$$

which can be solved by quadratures.

This completes the demonstration that in the general higher-dimensional rotating black hole spacetime (3.1) the reduced $(D-1)$-dimensional geodesic problem (2.11) allows the separation of the Hamilton-Jacobi equation (2.13) and therefore the stationary string configuration is completely integrable.

4. Hidden symmetries

The resulting complete integrability of the stationary string configuration in the Kerr-NUT-(A)dS spacetime (3.1) is connected with the existence of hidden symmetries of the $(D-1)$-dimensional effective metric H_{ij}. Namely, there exist $(n-1)$ irreducible Killing tensors $C^{ij}_{(k)}$, $(k = 1, \ldots, n-1)$, which give the constants of motion

$$C_k = C^{ij}_{(k)} p_i p_j \,, \quad D_{(m} C^{(k)}_{ij)} = 0 \,, \tag{4.1}$$

and allow the separation of the Hamilton-Jacobi equation (2.13) in the background H_{ij}. In the last formula $p_i = \partial_i S$ are the 'momenta' of geodesic motion and D_i denotes the covariant derivative with respect to H_{ij}.

One can easily find the explicit form of $C^{ij}_{(k)}$ by inverting (3.11). Let us multiply it by $A^{(l)}_\mu / U_\mu$, sum over μ, and use identities [22]:

$$\sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-1-k}}{U_\mu} A^{(l)}_\mu = \delta^l_k \,, \quad \sum_{\mu=1}^{n} \frac{A^{(k)}_\mu}{x_\mu^2 U_\mu} = \frac{A^{(k)}}{A^{(n)}} \,, \tag{4.2}$$

which are valid for $l, k = 0, \ldots, n-1$. Then we obtain

$$C^{ij}_{(k)} = K^{ij}_{(k)} - F_{(k)} H^{ij} \,, \tag{4.3}$$

$$F_{(k)} = \sum_{\mu=1}^{n} Q_\mu A^{(k)}_\mu - \frac{\varepsilon c A^{(k)}}{A^{(n)}} \,. \tag{4.4}$$

Here $K^{ij}_{(k)}$ are natural projections of the tensors

$$K^{ab}_{(k)} \partial_a \partial_b = \sum_{\mu=1}^{n} \frac{A^{(k)}_\mu}{Q_\mu U_\mu^2}\left(\sum_{l=0}^{m}(-x_\mu^2)^{n-1-l}\partial_{\psi_l}\right)^2 + \sum_{\mu=1}^{n} A^{(k)}_\mu Q_\mu (\partial_{x_\mu})^2 - \frac{\varepsilon A^{(k)}}{c A^{(n)}}(\partial_{\psi_n})^2 \,. \tag{4.5}$$

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That is, similar to (3.5), the direction ∂_{ψ_0} is projected out (the term $l=0$ is omitted).

In fact, the tensors $K^{ab}_{(k)}$, $(k=1,\dots,n-1)$, are the irreducible Killing tensors for the Kerr-NUT-(A)dS metric (3.1) [26, 22]. And so one can say that the hidden symmetries of the $(D-1)$-dimensional effective metric H_{ij} are 'inherited' from the hidden symmetries of g_{ab}.

A nontrivial property which follows from the separability of the Hamilton-Jacobi equation (see, e.g., [24]) is that the constants C_k mutually Poisson commute, or equivalently, the Schouten brackets, in the background H_{ij}, of the corresponding Killing tensors vanish:

$$\left[C_{(k)}, C_{(l)}\right]_H^{ijm} = C_{(k)}^{n(i} D_n C_{(l)}^{jm)} - C_{(l)}^{n(i} D_n C_{(k)}^{jm)} = 0\,. \tag{4.6}$$

Let us also mention that the projections $K^{ij}_{(k)}$ are the Killing tensors for the reduced metric h_{ij} and obey

$$\left[K_{(k)}, K_{(l)}\right]_h^{ijm} = 0\,. \tag{4.7}$$

These results can be easily obtained by separating the Hamilton-Jacobi equation in the background of the reduced metric h_{ij}. We expect them to be more general. (For a discussion and necessary conditions regarding the projection of a single Killing tensor see [16].)

We have seen that the existence of the Killing tensors $C^{ij}_{(k)}$ for the metric H_{ij} is the property inherited from the metric g_{ab} (3.1). This metric possesses even more fundamental symmetry — connected with the principal Killing-Yano tensor [28] from which all the Killing tensors (4.5) are derivable [26]. A natural question arises whether also H_{ij} admits any (not necessary principal) Killing-Yano tensor.

In a general case the answer is negative. The necessary conditions for a Killing tensor in 4D to be the 'square' of a Killing-Yano tensor were given by Collinson [29] (see also [30]). One can easily check that they are not satisfied and hence, at least in $4D$, the metric H_{ij} does not admit a Killing-Yano tensor. In higher dimensions we can exclude the existence of the 'special' principal Killing-Yano tensor for the metric H_{ij}.[2]

5. ξ-branes

In the above consideration we have focused on stationary strings, that is strings generated by a 1-parameter family of timelike Killing trajectories. There are two natural ways how one may try to generalize this construction. First, one may consider other Killing vector fields, and/or second, in the case when there exist more than one Killing vector, one may consider hypersurfaces formed by the set of Killing trajectories passing through the same 1-dimensional curve. Let us discuss these generalizations in more detail.

For simplicity we assume that the spacetime M^D allows p mutually commuting Killing vectors which we denote by $\xi^a_{(M)}$, $(M, N = 1,\dots,p)$. The Frobenius theorem implies that for each point of the spacetime M^D there exists (at least locally) a submanifold of dimension

[2]The special principal Killing-Yano tensor is a principal Killing-Yano tensor obeying the additional properties as defined in [31]. It was demonstrated in [32] that the only higher-dimensional spacetime admitting this special principal Killing-Yano tensor is the 'generalized' Kerr-NUT-AdS spacetime, i.e. the spacetime different from H_{ij}.

p generated by the Killing vectors $\xi^a_{(M)}$ passing through this point. In other words, the set $\xi = \{\xi^a_{(M)}\}$ defines a foliation of M^D. Similar to what was done in the Geroch formalism for one Killing vector field, one can define a quotient space S of M determined by the action of the isometry group generated by ξ. This generalization of the Geroch's formalism was developed in [33]. The metric g_{ab} of the spacetime M^D can be written as

$$g_{ab} = h_{ab} + \Xi_{ab}\,, \qquad h_{ab}\xi^a_{(M)} = 0\,, \qquad \Xi_{ab} = \sum_{M,N=1}^{p} a^{MN}\xi_{(M)a}\xi_{(N)b}\,. \tag{5.1}$$

Here a^{MN} is the $(p\times p)$ matrix which is inverse to the $(p\times p)$ matrix $a_{MN} = \xi_{(M)a}\xi^a_{(N)}$: $a^{MN}a_{NK} = \delta^M_K$. A tensor h_{ab} is a projection operator onto S.

Let us denote by y^i $(D-p)$ coordinates which are constant along the Killing surfaces generated by the set ξ, and by ψ^M the Killing parameters defined by the conditions

$$\xi^a_M\partial_a = \partial_{\psi^M}\,. \tag{5.2}$$

The metric g_{ab} in these coordinates $(x^a) = (y^i, \psi^M)$ takes the form

$$ds^2 = h_{ij}dy^i dy^j + \sum_{M,N=1}^{p} a^{MN}(\xi_{(M)a}dx^a)(\xi_{(N)b}dx^b)\,. \tag{5.3}$$

In these coordinates we also have

$$a_{MN} = \xi_{(M)a}\xi^a_{(N)} = \xi_{(N)M} = \xi_{(M)N}\,. \tag{5.4}$$

A natural generalization of stationary strings Σ_ξ are $(p+1)$-dimensional objects Σ^p_ξ which are formed by a 1-parameter family of Killing surfaces. We call them ξ-branes. In (y^i,ψ^M)-coordinates the equation of Σ^p_ξ is $y^i = y^i(\sigma)$. For this parametrization coordinates on Σ^p_ξ are $(\zeta^A) = (\psi^M,\sigma)$ $(A,B = 1,\dots,p+1)$. The induced metric on the ξ-brane takes the form

$$d\gamma^2 = \gamma_{AB}d\zeta^A d\zeta^B = (h+u)d\sigma^2 + 2d\sigma\sum_{M=1}^{p}\xi_{(M)\sigma}d\psi^M + \sum_{M,N=1}^{p} a_{MN}d\psi^M d\psi^N. \tag{5.5}$$

Here we have defined

$$h = h_{ij}\frac{dy^i}{d\sigma}\frac{dy^j}{d\sigma}\,, \quad \xi_{(M)\sigma} = \xi_{(M)i}\frac{dy^i}{d\sigma}\,, \quad u = \sum_{M,N=1}^{p} a^{MN}\xi_{(M)\sigma}\xi_{(N)\sigma}\,. \tag{5.6}$$

In order to derive (5.5) we used (5.4).

The metric γ_{AB} can be considered as a block matrix of the form

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{5.7}$$

where A is a 1-dimensional matrix and D is a matrix $(p \times p)$. If $|Z|$ is a determinant of a matrix Z, then one has the following relation for the determinant of a block matrix (see, e.g., [34])

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D||A - CD^{-1}B| . \tag{5.8}$$

Using this equation one obtains

$$\gamma = \det(\gamma_{AB}) = \begin{vmatrix} h+u & \xi_{(M)\sigma} \\ \xi_{(N)\sigma} & a_{MN} \end{vmatrix} = h\,\mathcal{F}_\xi\,, \tag{5.9}$$

where

$$\mathcal{F}_\xi = \det(a_{MN}) = \det(\xi^a_{(M)}\xi_{(N)a}) \tag{5.10}$$

is the Gram determinant for the set $\xi = \{\xi_{(M)}\}$ of the Killing vectors.

The Dirac-Nambu-Goto action for a $(p+1)$-dimensional brane is

$$I = -\mu \int d^{p+1}\zeta \sqrt{|\gamma|}\,, \tag{5.11}$$

where γ is the determinant of the induced metric on the brane γ_{AB}. For a ξ-brane this action reduces to the following expression[3]

$$I = -\mu V \mathcal{E}\,, \qquad dl^2 = h d\sigma^2\,, \tag{5.14}$$

$$V = \int d^p\psi^N\,, \qquad \mathcal{E} = \int \sqrt{\mathcal{F}_\xi} dl\,. \tag{5.15}$$

Thus after the dimensional reduction the problem of finding a configuration of a ξ-brane reduces to a problem of solving a geodesic equation in the reduced $(D-p)$-dimensional space with the metric

$$dH^2 = H_{ij}dy^i dy^j = \mathcal{F}_\xi h_{ij} dy^i dy^j\,. \tag{5.16}$$

If the original metric g_{ab} admits a Killing tensor K^{ab} then, since $h^{ij} = g^{ij}$, the natural projection K^{ij} is also a Killing tensor for the metric h_{ij}. However, the full effective metric H_{ij} does not inherit this symmetry unless the 'red-shift' factor $\mathcal{F}_\xi$ is of the special 'separable form'. Only then, the Hamilton-Jacobi equation (2.13) for the geodesic motion in the metric (5.16) allows complete separation of variables.

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[3] In our derivation we have focused on a 1-dimensional line in S generating ξ-branes. The same construction remains valid for, let us say, q-dimensional hyperspace in S in the case of a $(p+q)$-dimensional brane. Then, denoting coordinates on the worldvolume of such brane by $(\zeta^A) = (\psi^M, \sigma^\alpha)$, $(\alpha, \beta = 1, \dots, q)$, and repeating the same steps one would obtain

$$\gamma = \det(h_{\alpha\beta})F_\xi = hF_\xi, \quad h_{\alpha\beta} = h_{ij}\frac{dy^i}{d\sigma^\alpha}\frac{dy^j}{d\sigma^\beta}\,, \tag{5.12}$$

and

$$I = -\mu V \mathcal{E}, \quad \mathcal{E} = \int \sqrt{\mathcal{F}_\xi} dv, \quad dv = \sqrt{h} d^q\sigma\,. \tag{5.13}$$

6. ξ-branes in Kerr-NUT-AdS spacetime

6.1 Separability condition

Let us discuss now the problem of integrability of ξ-branes in the Kerr-NUT-(A)dS metric (3.1). There we have $m+1$ Killing fields ∂_{ψ_k}, $k = 0, \dots, m$ and we may choose any arbitrary subset of them as the set ξ. In general, however, the corresponding red-shift factor $\mathcal{F}_\xi$ will not be of the separable form.

More specifically, one requires that the red-shift factor can be written as

$$\mathcal{F}_\xi = \sum_{\mu=1}^{n} \frac{f_\mu(x_\mu)}{U_\mu}\,, \tag{6.1}$$

with f_μ functions of x_μ only, in order to allow the separation of variables for the Hamilton-Jacobi equation in the effective background H_{ij}. The corresponding Killing tensors ($k = 1, \dots, n-1$) would be then

$$C^{ij}_{(k)} = K^{ij}_{(k)} - f_{(k)} H^{ij}\,, \tag{6.2}$$

where $K^{ij}_{(k)}$ are due natural projections of (4.5), with directions from the set ξ projected out, and

$$f_{(k)} = \sum_{\mu=1}^{n} \frac{f_\mu A^{(k)}_\mu}{U_\mu}\,. \tag{6.3}$$

In the case of a stationary string, i.e. for $\xi = \{\partial_{\psi_0}\}$, the red-shift factor (3.6), the norm of the primary Killing field ∂_{ψ_0}, possesses the property (6.1), with

$$f_\mu = X_\mu - \frac{\epsilon c}{x_\mu^2}\,, \tag{6.4}$$

and the integrability proved in the section 3 is justified.

6.2 ξ-branes in 4D

In 4D a stationary string is the only nontrivial example of a ξ-brane for which (in these coordinates) integrability can be proved. Indeed, as discussed in [16] only in the exceptionally symmetric case of de Sitter space itself one can obtain the integrability of the axially symmetric ξ-string with $\xi = \{\partial_{\psi_1}\}$.[4]

The last possibility of a ξ-brane in 4D Kerr-NUT-(A)dS spacetime is the axially symmetric stationary domain wall, $\xi = \{\partial_{\psi_0}, \partial_{\psi_1}\}$. Let us consider this important example in more detail. The action takes the form

$$I = -\mu \Delta\psi_0 \Delta\psi_1 \mathcal{E}\,, \quad \mathcal{E} = \int d\sigma \sqrt{H_{ij} \frac{dy^i}{d\sigma} \frac{dy^j}{d\sigma}}\,, \tag{6.5}$$

[4]The asymmetry between the Killing fields is connected with the separability of the Klein-Gordon equation, see, e.g., [16] and reference therein. In higher-dimensional spacetime (3.1) this separability was demonstrated in [22].

where the effective 2-dimensional metric is

$$dH^2 = H_{ij}dy^i dy^j = \mathcal{F}_\xi \left(\frac{dx_1^2}{Q_1} + \frac{dx_2^2}{Q_2} \right). \tag{6.6}$$

The red-shift factor reads

$$\mathcal{F}_\xi = \begin{vmatrix} g_{\psi_0\psi_0} & g_{\psi_0\psi_1} \\ g_{\psi_0\psi_1} & g_{\psi_1\psi_1} \end{vmatrix} = \sum_{\mu=1}^{2} \frac{f_\mu}{U_\mu}\,, \tag{6.7}$$

where

$$f_\mu = x_\mu^2 X_\mu (X_1 + X_2). \tag{6.8}$$

Evidently, f_μ becomes function of x_μ only in the case when all parameters, but c_0, vanish. Only in that trivial case the Hamilton-Jacobi equation for the axially symmetric stationary domain wall in 4D can be separated.

The stationary string configuration remains the only one separable also in the standard Boyer-Lindquist coordinates which can be obtained from our coordinates by the identifications given in [35].

6.3 ξ-branes in 5D

In 5D the situation is more interesting. There we can prove the integrability of the axisymmetric ξ-string, $\xi = \{\partial_{\psi_1}\}$, under the condition that $c_1 = 0$. Indeed, then the red-shift factor takes the separable form (6.1) with

$$f_1(x_1) = 2b_2 x_1^4 + c x_1^2\,, \quad f_2(x_2) = 2b_1 x_2^4 + c x_2^2\,. \tag{6.9}$$

Also, the axially symmetric stationary ξ-brane, $\xi = \{\partial_{\psi_0}, \partial_{\psi_1}\}$ is completely integrable in the case of a vacuum ($c_2 = 0$) 5D spacetime (3.1) with $c_1 = 0$. In that case,

$$f_1(x_1) = 4b_1 b_2 x_1^2 + 2cb_1\,, \quad f_2(x_2) = 4b_1 b_2 x_2^2 + 2cb_2\,. \tag{6.10}$$

In both cases the nontrivial Killing tensor responsible for the integrability is given by (6.2).

However restrictive and unlikely to be generally satisfied the condition (6.1) seems, the above examples illustrate the special cases where complete integrability of ξ-branes can be analytically proved. We postpone the discussion of the existence of other nontrivial examples elsewhere.

7. Summary

We have studied integrability of the Nambu-Goto equations for a stationary string configuration near a higher-dimensional rotating black hole. In a general stationary spacetime this problem reduces to finding a geodesic in the effective $(D-1)$-dimensional background H_{ij}. In the Kerr-NUT-(A)dS spacetime (3.1) the geodesic equation can be integrated by separation of variables of the corresponding Hamilton-Jacobi equation. This separability is a consequence of the fact that H_{ij} inherits some of the hidden symmetries of the black

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hole. Namely, it inherits $(n-1)$ irreducible mutually commuting Killing tensors which correspond to natural projections of the Killing tensors present in g_{ab}. In a general case there are no (antisymmetric) Killing-Yano tensors generating these (symmetric rank 2) Killing tensors.

The problem of integrating of equations for ξ-branes is more complicated. We gave some examples where these equations are completely integrable, but in the general case the complete integrability is not possible. It would be interesting to find other, physically interesting, examples of completely integrable ξ-branes in higher dimensional black hole spacetimes. It is also interesting to study cases where there exist non-complete but non-trivial sets of (quadratic in momenta) integrals of motion for ξ-branes related to the hidden symmetries of the black hole background.

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